SUPREMUM OF PERELMAN'S ENTROPY AND KÄHLER-RICCI FLOW ON A FANO MANIFOLD

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Abstract. In this paper, we extend the method in [TZhu5] to study the energy level $L(\cdot)$ of Perelman's entropy $\lambda(\cdot)$ for Kähler-Ricci flow on a Fano manifold. Consequently, we first compute the supremum of $\lambda(\cdot)$ in Kähler class $2\pi c_1(M)$ under an assumption that the modified Mabuchi's K-energy $\mu(\cdot)$ defined in [TZhu2] is bounded from below. Secondly, we give an alternative proof to the main theorem about the convergence of Kähler-Ricci flow in [TZhu3].

Introduction

In this paper, we extend the method in [TZhu5] to study the energy level $L(\cdot)$ of Perelman's entropy $\lambda(\cdot)$ for Kähler-Ricci flow on an n-dimensional compact Kähler manifold (M, J) with positive first Chern class $c_1(M) > 0$ (namely called a Fano manifold). We will show that $L(\cdot)$ is independent of choice of initial Kähler metrics in $2\pi c_1(M)$ under an assumption that the modified Mabuchi's K-energy $\mu(\cdot)$ is bounded from below (cf. Proposition 3.1 in Section 3). The modified Mabuchi's K-energy $\mu(\cdot)$ is a generalization of Mabuchi's K-energy. It was showed in [TZhu2] that $\mu(\cdot)$ is bounded from below if M admits a Kähler-Ricci soliton.

As an application of Proposition 3.1, we first compute the supremum of Perelman's entropy $\lambda(\cdot)$ in Kähler class $2\pi c_1(M)$ [Pe]. More precisely, we prove that

Theorem 0.1. *Suppose that the modified Mabuchi's K-energy is bounded from below. Then*

$$\sup\{\lambda(g')|\ g' \in \mathcal{K}_X\} = (2\pi)^{-n}[nV - N_X(c_1(M))]. \tag{0.1}$$

Here the quantity $N_X(c_1(M))$ is a nonnegative invariance in $\mathcal{K}_X$ and it is zero iff the Futaki-invariant vanishes [Fu]. We denote $\mathcal{K}_X$ to be a class

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of K_X-invariant Kähler metrics in $2\pi c_1(M)$, where K_X is an one-parameter compact subgroup of holomorphisms transformation group generated by an extremal holomorphic vector field X for Kähler-Ricci solitons on M [TZhu2]. We note that we do not need to assume an existence of Kähler-Ricci solitons in Theorem 0.1. In fact, if we assume the existence of Kähler-Ricci solitons then we can use a more direct way to prove Theorem 0.1 and that the supremum of $\lambda(\cdot)$ can be achieved in $\mathcal{K}_X$ (cf. Section 1). It seems that the supremum of $\lambda(\cdot)$ can be achieved in the total space of Kähler potentials in $2\pi c_1(M)$ if M admits a Kähler-Ricci soliton. In a special case of small neighborhood of a Kähler-Ricci soliton the positivity has been verified by computing the second variation of $\lambda(\cdot)$ in [TZhu4].

As another application of Proposition 3.1, we prove the following convergence result for Kähler-Ricci flow.

Theorem 0.2. *Let (M, J) be a compact Kähler manifold which admits a Kähler-Ricci soliton (g_{KS}, X). Then Kähler-Ricci flow with any initial Kähler metric in $\mathcal{K}_X$ will converge to a Kähler-Ricci soliton in C^∞ in the sense of Kähler potentials. Moreover, the convergence can be made exponentially.*

We note that that without loss of generality we may assume that a Kähler-Ricci soliton g_{KS} on M is corresponding to the above X (cf. [TZhu1], [TZhu2]). Theorem 0.2 was first proved by Tian and Zhu in [TZhu3] by using an inequality of Moser-Trudinger type established in [CTZ][1]. Here we will modify arguments in [TZhu5] in our general case that (M, J) admits a Kähler-Ricci soliton to give an alternative proof of this theorem. This new proof does not use such an inequality of Moser-Trudinger type. Moreover, in particular, in case that (M, J) admits a Kähler-Einstein metric this new proof allows us to avoid to use a deep result recentlly proved by Chen and Sun in [CS] for the uniqueness of Kähler-Einsteins in the sense of orbit space to give a self-contained proof to the main theorem in [TZhu5].

The organization of paper is as follows. In Section 1, we discuss an upper bound of $\lambda(\cdot)$ in general case-without any condition for $\mu(\cdot)$ and show that the quantity $(2\pi)^{-n}[nV - N_X(c_1(M))]$ is an upper bound of $\lambda(\cdot)$ in $\mathcal{K}_X$ (cf. Proposition 1.4). In Section 2, we will summarize to give some estimates for modified Ricci potentials of evolved Kähler metrics along Kähler-Ricci flow (cf. Proposition 2.3). In Section 3, we prove Proposition 3.1 and so do Theorem 0.1. Theorem 0.2 will be proved in Section 6. In Section 4, we improve our key Lemma 3.2 in Section 3 independent of time t (cf. Proposition 4.2). Section 5 is a discussion about an upper bound of $\lambda(\cdot)$

[1] We need to add more details about how to use the Moser-Trudinger typed inequality in general case.

in $\mathcal{K}_Y$ for a general holomorphic vector field $Y \in \eta_r(M)$. Section 7 is an appendix where we discuss the gradient estimate and Laplace estimate for the minimizers of Perelman's W-functional along the Kähler-Ricci flow.

1. An upper bound of $\lambda(\cdot)$

In this section, we first review Perelman's W-functional for triples (g, f, τ) on a closed m-demensional Riemannian manifold M (cf. [Pe], [TZhu5]). Here g is a Riemannian metric, f is a smooth function and τ is a constant. In our situation, we will normalize volume of g by

$$\int_M dV_g \equiv V \tag{1.1}$$

and so we can fix τ by $\frac{1}{2}$. Then the W-functional depends only on a pair (g, f) and it can be reexpressed as follows:

$$W(g, f) = (2\pi)^{-m/2} \int_M [\frac{1}{2}(R(g) + |\nabla f|^2) + f]e^{-f} dV_g, \tag{1.2}$$

where $R(g)$ is a scalar curvature of g and (g, f) satisfies a normalization condition

$$\int_M e^{-f} dV_g = V. \tag{1.3}$$

Then Perelman's entropy $\lambda(g)$ is defined by

$$\lambda(g) = \inf_f \{W(g, f)|\ (g, f) \text{ satisfies (1.3)}\}.$$

It is well known that $\lambda(g)$ can be attained by some smooth function f (cf. [Ro]). In fact, such a f satisfies the Euler-Lagrange equation of $W(g, \cdot)$,

$$\triangle f + f + \frac{1}{2}(R - |\nabla f|^2) = (2\pi)^{m/2} V^{-1} \lambda(g). \tag{1.4}$$

Following Perelman's computation in [Pe], we can deduce the first variation of $\lambda(g)$,

$$\delta\lambda(g) = -(2\pi)^{-m/2} \int_M < \delta g, \mathrm{Ric}(g) - g + \nabla^2 f > e^{-f} dV_g, \tag{1.5}$$

where $\mathrm{Ric}(g)$ denotes the Ricci tensor of g and $\nabla^2 f$ is the Hessian of f. Hence, g is a critical point of $\lambda(\cdot)$ if and only if g is a gradient shrinking Ricci-soliton which satisfies

$$\mathrm{Ric}(g) + \nabla^2 f = g, \tag{1.6}$$

where f is a minimizer of $W(g, \cdot)$. The following lemma was proved in [TZhu5] for the uniqueness of solutions (1.4) when g is a gradient shrinking Ricci soliton.

Lemma 1.1. *If g satisfies (1.6) for some f, then any solution of (1.4) is equal to f modulo a constant. Consequently, a minimizer of $W(g,\cdot)$ is unique if the metric g is a gradient shrinking Ricci-soliton. Conversely, if f is a function in (1.6) for g, then f satisfies (1.4).*

In case that (M,J) is an n-dimensional Fano manifold, for any Kähler metric g in $2\pi c_1(M)$, (1.1) is equal to

$$\int_M dV_g = \int_M \omega_g^n = (2\pi)^n \int_M c_1(M)^n \equiv V. \tag{1.7}$$

Moreover, (1.6) becomes an equation for Kähler-Ricci solitons,

$$Ric(\omega_g) - \omega_g = \mathrm{L}_X \omega_g,$$

where $Ric(\omega_g)$ is a Ricci form of g and L_X denotes the Lie derivative along a holomorphic vector field X on M. By the uniqueness of Kähler-Ricci solitons [TZhu1], [TZhu2], we may assume that X lies in a reductive Lie subalgebra $\eta_r(M)$ of $\eta(M)$ after a holomorphism transformation, where $\eta(M)$ consists of all holomorphic vector fields on M. Such a X (we call it an extremal holomorphic vector field for Kähler-Ricci solitons) can be determined as follows.

Let $\mathrm{Aut}_r(M)$ be a connected Lie subgroup of automorphisms group of M generated by $\eta_r(M)$. Let K be a maximal compact subgroup of $\mathrm{Aut}_r(M)$. Without loss of generality, we may choose a K-invariant background metric g with its Kähler form ω_g in $2\pi c_1(M)$. In [TZhu2], as an obstruction to Kähler-Ricci solitons, Tian and Zhu introduced a modified Futaki-invariant $F_X(v)$ for any $X, v \in \eta(M)$ by

$$F_X(Z) = \int_M Z(h_g - \hat{\theta}_{X,\omega_g}) e^{\hat{\theta}_{X,\omega_g}} \omega_g^n, \quad \forall\ Z \in \eta(M), \tag{1.8}$$

where h_g is a Ricci potential of g and $\hat{\theta}_{X,\omega_g}$ is a real-valued potential of X associated to g defined by $L_X\omega_g = \sqrt{-1}\partial\bar{\partial}\hat{\theta}_{X,\omega_g}$ with a normalization condition

$$\int_M \hat{\theta}_{X,\omega_g} e^{h_g} \omega_g^n = 0. \tag{1.9}$$

It was showed that there exists a unique $X \in \eta_r(M)$ such that

$$F_X(v) \equiv 0,\ \forall\ v \in \eta_r(M).$$

Moreover, $F_X(v) \equiv 0$, for any $v \in \eta(M)$ if (M,J) admits a Kähler-Ricci soliton.

Let K_X be an one-parameter compact subgroup of holomorphisms transformation group generated by X. We denote $\mathcal{K}_X$ to be a class of K_X-invariant Kähler metrics in $2\pi c_1(M)$. Let θ_{X,ω_g} be a real-valued potential

of X associated to g with a normalization condition

$$\int_M e^{\theta_{X,\omega_g}}\omega_g^n = \int_M \omega_g^n = V. \tag{1.10}$$

Clearly, $\theta_{X,\omega_g} = \hat{\theta}_{X,\omega_g} - c_X$ for some constant c_X which is independent of $g \in \mathcal{K}_X$.

Definition 1.2. *For $g \in \mathcal{K}_X$, define $N_X(\omega_g)$ by*

$$N_X(\omega_g) = \int_M \theta_{X,\omega_g} e^{\theta_{X,\omega_g}}\omega_g^n.$$

By Jensen's inequality, it is easy to see

$$\begin{aligned}&\frac{1}{V}\int_M (-\theta_{X,\omega_g})e^{\theta_{X,\omega_g}}\omega_g^n\\ &\leq \log\{\frac{1}{V}\int_M e^{-\theta_{X,\omega_g}}e^{\theta_{X,\omega_g}}\omega_g^n\} = 0.\end{aligned}$$

The equality holds if and only if $\theta_{X,\omega_g} = 0$. This shows that $N_X(\omega_g)$ is nonnegative and it is zero if and only if the Futaki-invariant vanishes [Fu]. Moreover, we have

Lemma 1.3. *$N_X(\omega_g)$ is independent of choice of g in $\mathcal{K}_X$.*

Proof. Choose a K-invariant Kähler form ω in $2\pi c_1(M)$. Then for any Kähler metric g in $\mathcal{K}_X$ there exists a Kähler potential φ such that the imaginary part of $X(\varphi)$ vanishes and Kähler form of g satisfies

$$\omega_g = \omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi.$$

Thus we suffice to prove

$$N_X(\omega_\varphi) = N_X(\omega_{t\varphi}),\ \forall\ t \in\ [0,1],$$

where $\omega_{t\varphi} = \omega + t\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi$. This follows from

$$\begin{aligned}\frac{dN_X(\omega_{t\varphi})}{dt} &= \int_M X(\varphi)e^{\theta_{X,\omega_{t\varphi}}}\omega_{t\varphi}^n + \int_M \theta_{X,\omega_{t\varphi}}(X+\triangle)(\varphi)e^{\theta_{X,\omega_{t\varphi}}}\omega_{t\varphi}^n\\ &= \int_M X(\varphi)e^{\theta_{X,\omega_{t\varphi}}}\omega_{t\varphi}^n - \int_M \nabla^i\varphi\nabla_{\bar{i}}\theta_{X,\omega_{t\varphi}}\omega_{t\varphi}^n\\ &= 0.\end{aligned}$$

Here we have used the fact

$$\theta_{X,\omega_{t\varphi}} = \theta_{X,\omega_0} + tX(\varphi).$$

∎

By the above lemma, $N_X(\cdot)$ is an invariance on $\mathcal{K}_X$, which is independent of choice of g. For simplicity, we denote this invariance by $N_X(c_1(M))$. The following proposition gives an upper bound of $\lambda(\cdot)$ in $\mathcal{K}_X$ related to $N_X(c_1(M))$.

Proposition 1.4.

$$\sup_{g\in\mathcal{K}_X} \lambda(g) \le (2\pi)^{-n}[nV - N_X(c_1(M))].$$

Proof. Since $\lambda(g) \le W(g, -\theta_{X,\omega_g})$, we suffice to prove

$$W(g, -\theta_{X,\omega_g}) = (2\pi)^{-n}[nV - N_X(c_1(M))]. \tag{1.11}$$

In fact, by using the facts $R(g) = 2n + \Delta h_g$ and

$$\int_M (\Delta\theta_{X,\omega_g} + |\nabla\theta_{X,\omega_g}|^2)e^{\theta_{X,\omega_g}}\omega_g^n = 0,$$

we have

$$\begin{aligned}
&\int_M (R(g) + |\nabla\theta_{X,\omega_g}|^2)e^{\theta_{X,\omega_g}}\omega_g^n \\
&= 2nV + \int_M (\Delta h_g - \Delta\theta_{X,\omega_g})e^{\theta_{X,\omega_g}}\omega_g^n \\
&= 2nV - \int_M < \nabla(h_g - \theta_{X,\omega_g}), \nabla\theta_{X,\omega_g} > e^{\theta_{X,\omega_g}}\omega_g^n \\
&= 2nV - 2\int_M X(h_g - \theta_{X,\omega_g})e^{\theta_{X,\omega_g}}\omega_g^n \\
&= 2nV - 2e^{-c_X}F_X(X).
\end{aligned}$$

In the last equality above, we used the relation (1.8). Since X is extremal , we have

$$F_X(X) = 0.$$

Thus by (1.2) for $f = -\theta_{X,\omega}$ together with Lemma 1.3, one will get (1.11). $\square$

In case that M admits a Kähler-Ricci soliton g_{KS}, by Lemma 1.1, a minimizer f of $W(g_{KS}, \cdot)$ in $\mathcal{K}_X$ must be $-\theta_X$. Thus for any $g \in \mathcal{K}_X$, by Proposition 1.4, we have

$$\begin{aligned}
\lambda(g_{KS}) &= W(g_{KS}, -\theta_X) \\
&= (2\pi)^{-n}[nV - N_X(c_1(M))] \ge \lambda(g).
\end{aligned}$$

Therefore we get the following corollary.

Corollary 1.5. *Suppose that (M, J) admits a Kähler-Ricci soliton g_{KS}. Then g_{KS} is a global maximizer of $\lambda(\cdot)$ in $\mathcal{K}_X$ and*

$$\lambda(g_{KS}) = (2\pi)^{-n}[nV - N_X(c_1(M))]. \tag{1.12}$$

Remark 1.6. *Corollary 1.5 implies that a Kähler-Einstein metric is a global maximizer of $\lambda(\cdot)$ in $2\pi c_1(M)$ even with varying complex structures and supremum of $\lambda(\cdot)$ is $(2\pi)^{-n}nV$ since $N_X(c_1(M)) = 0$. Note that $N_X(c_1(M)) > 0$ if the Futaki-invariant does not vanish. Thus Corollary 1.5 also implies that the supremum of $\lambda(\cdot)$ in case that (M, J) admits a Kähler-Ricci soliton is strictly less than one in case that (M, J) admits a Kähler-Einstein metric.*

2. Estimates for modified Ricci potentials

In this section, we summarize some apriori estimates for modified Ricci potentials of evolved Kähler metrics along Kähler-Ricci flow. Some similar estimates have been also discussed in [TZhu3] and [PSSW], we refer the readers to those two papers. We consider the following (normalized) Kähler-Ricci flow:

$$\frac{\partial g(t, \cdot)}{\partial t} = -\mathrm{Ric}(g(t, \cdot)) + g(t, \cdot), \ g(0) = g, \tag{2.1}$$

where Kähler form of g is in $2\pi c_1(M)$. It was proved in [Ca] that (2.1) has a global solution $g_t = g(t, \cdot)$ for all time $t > 0$. For simplicity, we denote by $(g_t; g)$ a solution of (2.1) with initial metric g. Since the flow preserves the Kähler class, we may write Kähler form of g_t as

$$\omega_\phi = \omega_g + \sqrt{-1}\partial\overline{\partial}\phi$$

for some Käher potential $\phi = \phi_t$.

Let $X \in \eta_r(M)$ be the extremal holomorphic vector field on M as in Section 1 and $\sigma_t = \exp\{tX\}$ an one-parameter subgroup generated by X. Let $\phi' = \phi_{\sigma_t}$ be corresponding Kähler potentials of $\sigma_t^\star \omega_{\phi_t}$. Then $\omega_{\phi'}$ will satisfy a modified Kähler-Ricci flow,

$$\frac{\partial}{\partial t}\omega_{\phi'} = -\mathrm{Ric}(\omega_{\phi'}) + \omega_{\phi'} + L_X\omega_{\phi'}. \tag{2.2}$$

Equation (2.2) is equivalent to the following Monge-Ampére flow for ϕ' (modulo a constant),

$$\frac{\partial \phi'}{\partial t} = \log \frac{\omega_{\phi'}^n}{\omega_g^n} + \phi' + \theta_{X, \omega_{\phi'}} - h_g, \ \phi'(0, \cdot) = c, \tag{2.3}$$

where c is a constant and all Kähler potentials $\phi' = \phi'_t = \phi'(t, \cdot)$ are in a space given by

$$\mathcal{P}_X(M, \omega) = \{\varphi \in C^\infty(M) |\ \omega_\varphi = \omega + \sqrt{-1}\partial\overline{\partial}\varphi > 0, \quad \mathrm{Im}(X(\varphi)) = 0\}.$$

By using the maximum principle to (2.2) or (2.3), we get

$$h_{\phi'} - \theta_{X,\omega_{\phi'}} = -\frac{\partial}{\partial t}\phi' + c_t, \tag{2.4}$$

for some constants c_t. Here $h_{\phi'}$ are Ricci potentials of $\omega_{\phi'}$ which are normalized by

$$\int_M e^{h_{\phi'}}\omega_{\phi'}^n = V. \tag{2.5}$$

The following estimates are due to G. Perelman. We refer the readers to [ST] for their proof.

Lemma 2.1. *There are constants c and C depending only on the initial metric g such that (a) diam$(M,\omega_{\phi'}) \le C$; (b) vol$(B_r(p),\omega_{\phi'}) \ge cr^{2n}$; (c) $\|h_{\phi'}\|_{C^0(M)} \le C$; (d) $\|\nabla h_{\phi'}\|_{\omega_{\phi'}} \le C$; (e) $\|\Delta h_{\phi'}\|_{C^0(M)} \le C$.*

Recall that the modified Mabuchi's K-energy $\mu(\cdot)$ is defined in $\mathcal{P}_X(M,\omega)$ by

$$\begin{aligned}\mu(\varphi) = & -\frac{n}{V}\int_0^1\int_M \dot{\psi}[\mathrm{Ric}(\omega_\psi) - \omega_\psi - \sqrt{-1}\partial\overline{\partial}\theta_{X,\omega_\psi} \\ & + \sqrt{-}\overline{\partial}(h_{\omega_\psi} - \theta_{X,\omega_\psi}) \wedge \partial\theta_{X,\omega_\psi})] \wedge e^{\theta_{X,\omega_\psi}}\omega_\psi^{n-1} \wedge dt,\end{aligned}$$

where $\psi = \psi_t$ $(0 \le t \le 1)$ is a path connecting 0 to φ in $\mathcal{P}_X(M,\omega)$. If $X = 0$, then $\mu_{\omega_g}(\phi)$ is nothing but Mabuchi's K-energy [Ma]. Then by (2.2), we have

$$\frac{d\mu(\phi')}{dt} = -\frac{1}{V}\int_M \|\overline{\partial}\frac{\partial\phi'}{\partial t}\|^2_{\omega_{\phi'}} e^{\theta_{X,\omega_{\phi'}}}(\omega_{\phi'})^n \le 0. \tag{2.6}$$

This implies that $\mu(\phi')$ is uniformly bounded if $\mu(\cdot)$ is bounded from below in $\mathcal{P}_X(M,\omega)$.

Let $u_{X,\phi'} = u_{X,\omega_{g_t'}} = h_{\phi'} - \theta_{X,\omega_{\phi'}}$. Then

Lemma 2.2. *There exists a uniform C such that*

$$\|\nabla u_{X,\phi'}\|_{\omega_{\phi'}} \le C.$$

Proof. First we note that $\theta_{X,\omega_{\phi'}}$ is uniformly bounded in $\mathcal{P}_X(M,\omega)$ (cf. [Zhu1], [ZZ]). Then by (c) of Lemma 2.1, we have

$$\|u_{X,\phi'}\|_{C^0} = \|u_{X,\omega_{g_s'}}\|_{C^0} \le C, \ \forall\ s > 0$$

for some uniform constant C. Now we consider the flow (2.3) with zero as an initial Kähler potential and the background Kähler form ω_g replaced by $\omega_{g_s'}$. By an estimate in Lemma 4.3 in [CTZ], we see

$$t\|\nabla u_{X,\omega_{g'_{s+t}}}\|^2_{\omega_{g'_{s+t}}} \le e^{2t}\|u_{X,\omega_{g_s'}}\|_{C^0}, \ \forall\ t > 0.$$

In particular, we get

$$\|\nabla u_{X,\omega_{g'_{s+t}}}\|^2_{\omega_{g'_{s+t}}} \le C', \ \forall\, t \in [1,2].$$

Since the above estimate is independent of s, we conclude that the lemma is true.

□

Now we begin to prove the main result in this section.

Proposition 2.3. *Suppose that $\mu(\cdot)$ is bounded from below in $\mathcal{P}_X(M,\omega)$. Then we have:*
(a) $\lim_{t\to\infty} ||u_{X,\phi'}||_{C^0} = 0$;
(b) $\lim_{t\to\infty} ||\nabla u_{X,\phi'}||_{\omega_{\phi'}} = 0$;
(c) $\lim_{t\to\infty} ||\triangle u_{X,\phi'}||_{C^0} = 0$.

Proof. Let $H(t) = \int_M |\nabla u_{X,\omega'_{g_t}}|^2 e^{\theta_{X,\omega'_{g_t}}} \omega^n_{g'_t}$. Then by (2.6), one sees that there exists a sequence of $t_i \in [i, i+1]$ such that

$$\lim_{i\to\infty} H(t_i) = 0.$$

Thus by using a differential inequality

$$\frac{dH(t)}{dt} \le CH(t),$$

where C is a uniform constant (cf. [PSSW]), we get

$$\lim_{t\to\infty} \int_M |\nabla u_{X,\omega_{g'_t}}|^2_{g'_t} e^{\theta_{X,\omega_{g'_t}}} \omega^n_{g'_t} = 0. \tag{2.7}$$

Let

$$\tilde{u}_t = u_{X,\omega_{g'_t}} - \frac{1}{V}\int_M u_{X,\omega_{g'_t}} e^{h'_t} \omega^n_{g'_t},$$

where $h'_t = h_{\phi'(t,\cdot)}$. Then by using the weighted Poincaré inequality in [TZhu3] together with (c) of Lemma 2.1, we obtain from (2.7),

$$\int_M \tilde{u}^2_t e^{h'_t} \omega^n_{g'_t} \le \int_M |\nabla u_{X,\omega_{g'_t}}|^2_{g'_t} e^{h'_t} \omega^n_{g'_t} \to 0, \ \text{as} \ \ t\to\infty.$$

Consequently, we derive

$$\lim_{t\to\infty} \int_M \tilde{u}^2_t \omega^n_{g'_t} = 0. \tag{2.8}$$

We claim

$$\lim_{t\to\infty} \|\tilde{u}_t\|_{C^0} = 0.$$

The claim immediately implies (a) of Proposition 2.3 by the normalization conditions

$$\int_M e^{\theta_{X,\omega_{g'_t}}} \omega^n_{g'_t} = \int_M e^{h'_t} \omega^n_{g'_t} = V.$$

To prove the claim, we need to use an inequality

$$\|\tilde{u}_t\|_{C^0}^{n+1} \leq C\|\nabla u_{X,\omega_{g'_t}}\|_{g'_t}^n [\int_M \tilde{u}_t^2 \omega_{g'_t}^n]^{\frac{1}{2}}. \tag{2.9}$$

(2.9) can be proved by using the non-collapsing estimate (b) in Lemma 2.1 (cf. [PSSW], [Zhu2]). Thus by Lemma 2.2 and (2.8), the claim is proved.

By (a) we can show that after a suitable choice of constant c in the flow (2.3) it holds

$$\lim_{t\to\infty} \|\frac{\partial}{\partial t}\phi'\|_{C^0} = 0.$$

In fact under the assumption of lower bound of modified K-energy, one can choose such a c (cf. [TZhu3]) such that

$$\lim_{t\to\infty} \int_M \frac{\partial}{\partial t}\phi' e^{\theta_{X,\omega_{\phi'}}} \omega_{\phi'}^n = 0.$$

Then by (2.4), we will get the conclusion. On the other hand, by Lemma 2.2 and (d) of Lemma 2.1, we have

$$\sup_{t\in[0,\infty)} \|X\|_{g'_t} < C$$

for some uniform constant C. Therefore, by using the following lemma we prove (b) and (c).

□

Lemma 2.4. *([PSSW]) There exist $\delta, K > 0$ depending only on n and the constant $C_X = \sup_{t\in[0,\infty)} \|X\|_{g'_t}$ with the following property. For ϵ with $0 < \epsilon \leq \delta$ and any $t_0 > 0$, if*

$$\|\frac{\partial \phi'}{\partial t}\|_{C^0}(t_0) \leq \epsilon,$$

then

$$\|\nabla u_{X,\omega_{g'_{t_0+2}}}\|_{g'_{t_0+2}}^2 + \|\Delta u_{X,\omega_{g'_{t_0+2}}}\|_{C^0} \leq K\epsilon.$$

3. Proof of Theorem 0.1

According to [TZhu5], an energy level $L(g)$ of entropy $\lambda(\cdot)$ along Kähler-Ricci flow $(g_t; g)$ is defined by

$$L(g) = \lim_{t\to\infty} \lambda(g_t).$$

By the monotonicity of $\lambda(g_t)$, we see that $L(g)$ exists and it is finite. In this section, our goal is to prove

Proposition 3.1. *Suppose that the modified Mabuchi's K-energy is bounded from below in $\mathcal{K}_X$. Then for any $g \in \mathcal{K}_X$.*

$$L(g) = (2\pi)^{-n}(nV - N_X(c_1(M)). \tag{3.1}$$

The above proposition shows that the energy level $L(g)$ of entropy $\lambda(\cdot)$ does not depend on the initial Kähler metric $g \in \mathcal{K}_X$. Thus by using the Kähler-Ricci flow $(g_t; g)$ for any Kähler metric $g \in \mathcal{K}_X$ and the monotonicity of $\lambda(g_t)$, we will get Theorem 0.1.

To prove Proposition 3.1, we need the following key lemma.

Lemma 3.2. *Let f_t be a minimizer of $W(g_t, \cdot)$-functional associated evolved Kähler metric g_t of (2.1) at time t and h_t a Ricci potential of g_t which satisfying the normalization (2.5) . Then there exists a sequence of $t_i \in [i, i+1]$ such that*
(a) $\lim_{t_i\to\infty} \|\Delta(f_{t_i} + h_{t_i})\|_{L^2(M,\omega_{g_{t_i}})} = 0$;
(b) $\lim_{t_i\to\infty} \|\nabla(f_{t_i} + h_{t_i})\|_{L^2(M,\omega_{g_{t_i}})} = 0$;
(c) $\lim_{t_i\to\infty} \|f_{t_i} + h_{t_i}\|_{C^0} = 0$.

Proof. Lemma 3.2 is a generalization of Proposition 4.4 in [TZhu5]. We will follow the argument there. First by (1.5), it is easy to see that

$$\frac{d}{dt}\lambda(g_t) = (2\pi)^{-n}\int_M |\mathrm{Ric}(g_t) - g_t + \nabla^2 f_t|^2_{g_t} e^{-f_t}\omega^n_{g_t}.$$

It follows that

$$\frac{d}{dt}\lambda(g_t)) \geq (2\pi)^{-n}\frac{1}{2n}\int_M |\triangle(h_t + f_t)|^2 e^{-f(t)}\omega^n_{g_t}.$$

Since $\lambda(g_t) \leq W(g_t, 0) = (2\pi)^{-n} nV$ are uniformly bounded, we see that there exists a sequence of $t_i \in [i, i+1]$ such that

$$\lim_{i\to\infty}\int_M |\triangle(h_{t_i} + f_{t_i})|^2 e^{-f_{t_i}}\omega^n_{g_{t_i}} = 0.$$

Note that f_t is uniformly bounded [TZhu5]. Hence we see that that (a) of the lemma is true. By (a), we also get

$$\begin{aligned} &\lim_{t_i\to\infty} \|\nabla(f_{t_i} + h_{t_i})\|_{L^2(M,\omega_{g_{t_i}})} \\ &\leq \lim_{t_i\to\infty}\int_M |f_{t_i} + h_{t_i}||\triangle(f_{t_i} + h_{t_i})|\omega^n_{g_{t_i}} \\ &\leq C\lim_{t_i\to\infty} \|\Delta(f_{t_i} + h_{t_i})\|_{L^2(M,\omega_{g_{t_i}})} = 0. \end{aligned} \tag{3.2}$$

This proves (b) of the lemma. It remains to prove (c).

Let $q_t = f_t + h_t$. Then

$$\begin{aligned} -\Delta q_t &= -\Delta f_t - \Delta h_t \\ &= f_t + \frac{1}{2}(R - |\nabla f_t|^2) - (2\pi)^{2n} V^{-1} \lambda(g_t) - \Delta h_t \leq C. \end{aligned} \tag{3.3}$$

Define

$$\tilde{q}_t = q_t - \frac{1}{V} \int_M q_t e^{h_t} \omega_{g_t}^n.$$

By using the weighted Poincaré inequality (cf. [TZhu3]), we have

$$\int_M \tilde{q_t}^2 e^{h_t} \omega_{g_t}^n \leq \int_M |\nabla q_t|^2 e^{h_t} \omega_{g_t}^n.$$

It follows by (b),

$$\lim_{i\to\infty} \int_M \tilde{q}_{t_i}^2 \omega_{g_{t_i}}^n = 0. \tag{3.4}$$

Hence, following an argument in the proof of Proposition 4.4 in [TZhu5], we will get estimates

$$\|\tilde{q_{t_i}}^+\|_{C^0} \leq C \|\tilde{q}_{t_i}\|_{L^2(M, \omega_{g_{t_i}})} \to 0, \text{ as } i \to \infty, \tag{3.5}$$

and

$$\lim_{i\to\infty} \int_M \tilde{q_{t_i}}^- \omega_{g_{t_i}}^n = 0, \tag{3.6}$$

where $q_t^+ = \max\{q_t, 0\}$ and $q_t^- = \min\{q_t, 0\}$. Consequently, we derive

$$\int_M \tilde{q_{t_i}} e^{-f_{t_i}} \omega_{g_{t_i}}^n = 0.$$

This implies

$$\lim_{i\to\infty} \int_M q_{t_i} e^{-f_{t_i}} \omega_{g_{t_i}}^n = 0 \tag{3.7}$$

according to the normalization $\int_M e^{-f_t} \omega_{g_t}^n = \int_M e^{h_t} \omega_{g_t}^n = V$.

Next we improve that

$$\lim_{i\to\infty} |q_{t_i}| = 0. \tag{3.8}$$

Let $u_t = e^{-\frac{f_t}{2}} - e^{\frac{h_t}{2}}$. We claim

$$\lim_{i\to\infty} \|u_{t_i}\|_{L^2(M, \omega_{g_{t_i}})} = 0. \tag{3.9}$$

In fact, by Jensen's inequality and (3.7), one sees

$$\begin{aligned} \frac{1}{V} \int_M e^{-\frac{f_{t_i}}{2}} e^{\frac{h_{t_i}}{2}} \omega_{g_{t_i}}^n &= \frac{1}{V} \int_M e^{\frac{f_{t_i} + h_{t_i}}{2}} e^{-f_{t_i}} \omega_{g_{t_i}}^n \\ &\geq e^{\frac{1}{2V} \int_M (f_{t_i} + h_{t_i}) e^{-f_{t_i}} \omega_{g_{t_i}}^n} \to 1, \text{ as } i \to \infty. \end{aligned}$$

On the other hand,

$$\int_M e^{-\frac{f_t}{2}} e^{\frac{h_t}{2}} \omega_{g_t}^n \leq (\int_M e^{-f_t}\omega_{g_t}^n)^{\frac{1}{2}} (\int_M e^{h_t}\omega_{g_t}^n)^{\frac{1}{2}} = V.$$

Hence

$$\lim_{i\to\infty} \int_M e^{-\frac{f_{t_i}}{2}} e^{\frac{h_{t_i}}{2}} \omega_{g_{t_i}}^n = V.$$

It follows

$$\lim_{i\to\infty} \int_M u_{t_i}^2 \omega_{g_{t_i}}^n = 2V - 2 \lim_{i\to\infty} \int_M e^{-\frac{f_{t_i}}{2}} e^{\frac{h_{t_i}}{2}} \omega_{g_{t_i}}^n = 0.$$

This completes the proof of claim.

Since equation (1.4) is equivalent to

$$\Delta v_t - \frac{1}{2} f_t v_t - \frac{1}{4} R(g_t) v_t = \frac{1}{2V}(2\pi)^n \lambda(g_t) v_t, \tag{3.10}$$

where $v_t = e^{\frac{-f_t}{2}}$, by Lemma 2.1, it is easy to see

$$|\Delta u_t| \leq C.$$

Then by the standard Moser's iteration, we get from (3.9),

$$\|u_{t_i}\|_{C^0} \leq C \|u_{t_i}\|_{L^2(M,\omega_{g_{t_i}})} \to 0, \text{ as } i \to \infty.$$

This implies (3.8), so we obtain (c) of the lemma.

□

Proof of Proposition 3.1. Note that $\frac{R(g_t)}{2} = n + \frac{1}{2}\Delta h_t$, where Δ is the Beltrima-Laplacian operator associated to the Riemannian metric g_t. Then

$$\int_M \frac{1}{2}(R(g_t) + |\nabla f_t|^2) e^{-f_t} dV_{g_t} = nV + \frac{1}{2}\int_M \Delta(f_t + h_t) e^{-f_t} dV_{g_t}.$$

Thus by (a) of Lemma 3.2, one sees that there exists a sequence of time t_i such that

$$\lim_{i\to\infty} \int_M \frac{1}{2}(R(g_{t_i}) + |\nabla f_{t_i}|^2) e^{-f_{t_i}} dV_{g_{t_i}} = nV. \tag{3.11}$$

On the other hand, since the modified Mabuchi's K-energy is bounded from below, we see that (a) of Proposition 2.3 is true. Then by (c) of Lemma 3.2, it follows

$$\lim_{i\to\infty} \|f_{t_i} + \theta_{X,\omega_{g_{t_i}}}\|_{C^0} = 0. \tag{3.12}$$

Here we used a fact $\sigma_t^\star \theta_{X,\omega_{g_t}} = \theta_{X,\omega_{g'_t}}$ since X lies in the center of $\eta_r(M)$ [TZhu1]. Hence

$$\begin{aligned}&\lim_{i\to\infty}\int_M f_{t_i}e^{-f_{t_i}}dV_{g_{t_i}}\\ &= -\lim_{i\to\infty}\int_M \theta_{X,\omega_{g_{t_i}}} e^{\theta_{X,\omega_{g_{t_i}}}}\omega_{g_{t_i}}^n = -N_X(c_1(M)).\end{aligned} \tag{3.13}$$

By combining (3.11) and (3.13), we get

$$\begin{aligned}&\lim_{i\to\infty}\lambda(g_{t_i}) = \lim_{i\to\infty}\int_M [\frac{1}{2}(R(g_{t_i}) + |\nabla f_{t_i}|^2) + f_{t_i}]e^{-f_{t_i}}dV_{g_{t_i}}\\ &= nV - N_X(c_1(M))\end{aligned}$$

Therefore, by using the monotonicity of $\lambda(g_t)$ along the flow $(g_t; g)$, we obtain (3.1).

□

It was showed in [TZhu4] that a Kähler-Ricci soliton is a local maximizer of $\lambda(\cdot)$ in the Kähler class $2\pi c_1(M)$. Together with Corollary 1.5, one may guess that a Kähler-Ricci soliton is a global maximizer of $\lambda(\cdot)$. More general, according to Theorem 0.1 , we propose the following conjecture.

Conjecture 3.3. *Suppose that the modified Mabuchi's K-energy is bounded from below. Then*

$$\sup_{\omega_{g'}\in 2\pi c_1(M)} \lambda(g') = (2\pi)^{-n}[nV - N_X(c_1(M))].$$

4. Improvement of Lemma 3.2

In this section, we use Perelman's backward heat flow to improve estimate (c) in Lemma 3.2 independent of t. Moreover, we show the gradient estimate of $f_t + h_t$ also holds. Although Lemma 3.2 is sufficient to be applied to prove Theorem 0.1 and Theorem 0.2, results of this section are independent of interests. We hope that these results will have applications in the future.

Fix any $t_0 \geq 1$. We consider a backward heat equation in $t \in [t_0 - 1, t_0]$,

$$\frac{\partial}{\partial t} f_{t_0}(t) = -\triangle f_{t_0}(t) + |\nabla f_{t_0}(t)|^2 - \triangle h_t \tag{4.1}$$

with an initial $f_{t_0}(t_0) = f_{t_0}$. Clearly, the equation preserves the normalizing condition $\frac{1}{V}\int_M e^{-f_{t_0}(t)}\omega_{g_t}^n = 1$. Moreover, by the maximum principle, we have

$$\|f_{t_0}(t)\|_{C^0} \leq C(g), \quad \forall\ t \in [t_0 - 1, t_0], \tag{4.2}$$

since $\triangle h_t$ are uniformly bounded. Here the constant $C(g)$ depends only on the initial metric g of (2.1).

Similarly to (1.5), we can compute

$$\begin{aligned}&\frac{d}{dt}W(g_t, f_{t_0})\\ &= (2\pi)^{-n}\int_M (\|\partial\overline{\partial}(h_t + f_{t_0}(t))\|^2 + \|\partial\partial f_{t_0}(t)\|^2)e^{-f_{t_0}(t)}\omega_{g_t}^n.\end{aligned} \tag{4.3}$$

By using (4.3), we want to prove

Lemma 4.1.

$$\|f_t + h_t - c_t\|_{L^2(M,g_t)} \to 0, \ \ as\ t \to \infty, \tag{4.4}$$

where $c_t = \frac{1}{V}\int_M (f_t + h_t)e^{h_t}\omega_{g_t}^n$.

Proof. First by (4.3), one sees

$$\begin{aligned}\lambda(g_{t_0}) - \lambda(g_{t_0-1}) &\geq W(g_{t_0}, f_{t_0}(t_0)) - W(g_{t_0-1}, f_{t_0}(t_0 - 1))\\ &\geq (2\pi)^{-n}\frac{1}{2n}\int_{t_0-1}^{t_0}\int_M |\triangle(f_{t_0}(t) + h_t)|^2 e^{-f_{t_0}(t)}\omega_{g_t}^n dt.\end{aligned}$$

It follows

$$\int_{t_0-1}^{t_0}\int_M |\triangle(f_{t_0}(t) + h_t)|^2\omega_{g_t}^n dt \to 0, \text{ as } \ t_0 \to \infty.$$

Thus by using the weighted Poincaré inequality as in (3.4) in last section , we will get

$$\begin{aligned}&\int_{t_0-1}^{t_0} dt\int_M (f_{t_0}(t) + h_t - c_{t_0}(t))^2\omega_{g_t}^n\\ &\leq C(g_0)[\int_{t_0-1}^{t_0} dt\int_M |\triangle(f_{t_0}(t) + h_t)|^2\omega_{g_t}^n]^{1/2} \to 0, \text{ as } t_0 \to \infty,\end{aligned} \tag{4.5}$$

where $c_{t_0}(t) = \frac{1}{V}\int_M (f_{t_0}(t) + h_t)e^{h_t}\omega_{g_t}^n$.

Next, since $\frac{dh_t}{dt} = \Delta h_t + h_t - a_t$, where $a_t = \frac{1}{V}\int_M h_t e^{h_t}\omega_{g_t}^n$, by a straightforward calculation, we see

$$\begin{aligned}&\frac{d}{dt}\int_M (h_t + f_{t_0} - c_{t_0}(t))^2\omega_{g_t}^n\\ &= \int_M [2(h_t + f_{t_0}(t) - c_{t_0}(t))(\triangle f_{t_0} - |\nabla f_{t_0}(t)|^2 + h_t - a_t - \frac{dc_{t_0}}{dt})\\ &- (h_t + f_{t_0}(t) - c_{t_0}(t))^2\triangle h_t]\omega_{g_t}^n\end{aligned}$$

Then by Lemma 3.2, we get

$$
\begin{aligned}
&|\frac{d}{dt}\int_M (h_t + f_{t_0}(t) - c_{t_0})^2 \omega_{g_t}^n| \\
&\le C + C\int_M \big(|\triangle f_{t_0}(t)| + |\nabla f_{t_0}(t)|^2 + |\frac{dc_{t_0}(t)}{dt}|\big)\omega_{g_t}^n \\
&\le C + C\int_M \big(|\nabla\bar\nabla(f_{t_0}(t)+h_t)|^2 + |\frac{dc_{t_0}(t)}{dt}|\big)\omega_{g_t}^n.
\end{aligned}
$$

Notice that

$$\frac{dc_{t_0}}{dt} = \frac{1}{V}\int_M [\triangle f_{t_0}(t) - |\nabla f_{t_0}(t)|^2 - (h_t + f_{t_0}(t))(h_t + a_t)]e^{h_t}\omega_{g_t}^n.$$

We can also estimate

$$|\frac{dc_{t_0}}{dt}| \le C + C\int_M |\nabla\bar\nabla(f_{t_0}(t)+h_t)|^2\omega_{g_t}^n.$$

Hence we derive

$$|\frac{d}{dt}\int_M (f_{t_0}(t) + h_t - c_{t_0}(t))^2 dv\big| \le C + C\int_M |\nabla\bar\nabla(f_{t_0}(t)+h_t)|^2\omega_{g_t}^n. \tag{4.6}$$

Therefore, according to

$$
\begin{aligned}
&\int_{t_0-1}^{t_0} dt\int_M |\nabla\bar\nabla(f_{t_0}(t)+h_t)|^2 e^{-f_{t_0}(t)}\omega_{g_t}^n \\
&\le (2\pi)^n(\lambda(g_{t_0}) - \lambda(g_{t_0-1})) \to 0,\ \text{as } t_0\to\infty,
\end{aligned}
$$

(4.5) and (4.6) will implies

$$\|f_{t_0}(t) + h_t - c_{t_0}(t)\|_{L^2(g_t,M)} \to 0,\ \text{as } t_0\to\infty,\ \forall\ t\in[t_0-1,t_0].$$

Consequently, we get (4.4). □

Proposition 4.2.

$$\|f_t + h_t\|_{C^0} + \|\nabla(f_t+h_t)\|_{g_t} \to 0, \quad as\ t\to\infty. \tag{4.7}$$

Proof. With the help of Lemma 4.1, by using same argument in the proof of (c) in Lemma 3.2, we can prove that

$$\|f_t + h_t\|_{C^0} \to 0, \quad \text{as } t\to\infty. \tag{4.8}$$

So we suffice to prove

$$\|\nabla(f_t+h_t)\|_{g_t} \to 0, \quad \text{as } t\to\infty. \tag{4.9}$$

We will use the Moser's iteration to obtain (4.9) as in lemma 7.2 in Appendix. We note by (4.8) and Theorem 7.1 that

$$\int_M |\nabla q_t|^2\omega_{g_t}^n = |\int_M -\Delta(f_t+h_t)(f_t+h_t)\omega_{g_t}^n| \le C\|f_t+h_t\|_{C^0} \to 0, \tag{4.10}$$

where $q_t = f_t + h_t$ satisfies an equation

$$\triangle q_t = \frac{1}{2}(|\nabla f_t|^2 - 2f_t + \triangle h_t) + (2\pi)^n V^{-1}\lambda(g_t) - n.$$

Let $w_t = |\nabla q_t|^2$. Then by the Bochner formula, we have

$$\triangle w_t = |\nabla\nabla q_t|^2 + |\nabla\bar{\nabla} q_t|^2 + \nabla_i \triangle q_t \nabla_{\bar{i}} q_t + \nabla_{\bar{i}} \triangle q_t \nabla_i q_t + R_{i\bar{j}} \nabla_{\bar{i}} q_t \nabla_j q_t.$$

Hence for any $p \geq 2$, it follows

$$\begin{aligned} &\frac{4(p-1)}{p^2}\int_M |\nabla w_t^{p/2}|^2 \omega_{g_t}^n = -\int_M w_t^{p-1} \triangle q_t \omega_{g_t}^n \\ &= -\int_M w_t^{p-1}(|\nabla\nabla q_t|^2 + |\nabla\bar{\nabla} q_t|^2)\omega_{g_t}^n \\ &- 2\mathrm{Re}\int_M q_t^{p-1} \nabla_i \triangle q_t \nabla_{\bar{i}} q_t \omega_{g_t}^n - \int_M q_t^{p-1} R_{i\bar{j}} \nabla_{\bar{i}} q_t \nabla_j q_t \omega_{g_t}^n. \end{aligned} \tag{4.11}$$

On the other hand, by Lemma 2.1 and Theorem 7.1, we estimate

$$\begin{aligned} &- 2\mathrm{Re}\int_M w_t^{p-1} \nabla_i \triangle q_t \nabla_{\bar{i}} w_t \omega_{g_t}^n \\ &= -\mathrm{Re}\int_M w_t^{q-1} \nabla_i (|\nabla f_t|^2 - 2f + \triangle h_t)\nabla_{\bar{i}} q_t \omega_{g_t}^n \\ &= -\mathrm{Re}\int_M (|\nabla f_t|^2 - 2f_t + \triangle h_t)(\frac{2(p-1)}{p} w_t^{\frac{p}{2}-1} \nabla_i w_t^{p/2} \nabla_{\bar{i}} q_t + w_t^{q-1}\triangle q_t)\omega_{g_t}^n \\ &\leq C(g)[\int_M \frac{2(p-1)}{p} w_t^{\frac{p-1}{2}} |\nabla w_t^{p/2}|\omega_{g_t}^n + \int_M w_t^{p-1}|\triangle q_t|]\omega_{g_t}^n \\ &\leq \frac{p-1}{p^2}\int_M |\nabla w^{p/2}|^2 \omega_{g_t}^n + C(g)' p \int_M w^{p-1}\omega_{g_t}^n \end{aligned}$$

and

$$\begin{aligned} &-\int_M w_t^{p-1} R_{i\bar{j}} \nabla_{\bar{i}} q_t \nabla_j q_t \omega_{g_t}^n \\ &= -\int_M w_t^p \omega_{g_t}^n - \int_M w_t^{p-1} \nabla_i \nabla_{\bar{j}} h_t \nabla_{\bar{i}} q_t \nabla_j q_t \omega_{g_t}^n \\ &= -\int_M w_t^p \omega_{g_t}^n + \int_M w_t^{p-1} \nabla_{\bar{j}} h_t (\nabla_{\bar{i}} q_t \nabla_i \nabla_j q_t \omega_{g_t}^n + \triangle q_t \nabla_j q_t)\omega_{g_t}^n \\ &+ \frac{2(p-1)}{p}\int_M w_t^{\frac{p}{2}-1} \nabla_i w_t^{p/2} \nabla_{\bar{j}} h_t \nabla_{\bar{i}} q_t \nabla_j q \omega_{g_t}^n \\ &\leq \int_M w_t^{p-1}(|\nabla\nabla q_t|^2 + \frac{1}{2}|\nabla\bar{\nabla} q_t|^2)\omega_{g_t}^n + \frac{p-1}{p^2}\int_M |\nabla w_t^{p/2}|^2 \omega_{g_t}^n \\ &+ C(g)(p-1)\int_M w_t^p \omega_{g_t}^n. \end{aligned}$$

Then substituting the above two inequalities into (4.11), we get

$$\int_M |\nabla w_t^{p/2}|^2 \omega_{g_t}^n \le C(g)(p-1)^2 \int_M w_t^{p-1}\omega_{g_t}^n, \ \forall\, p \ge 2.$$

By using Zhang's Sobolev inequality [Zha], we deduce

$$\Big(\int_M w_t^{p\nu}\Big)^{1/\nu}\omega_{g_t}^n \le C(g)C_s(q-1)^2 \int_M w_t^{p-1}\omega_{g_t}^n, \quad \forall\, p \ge 2, \tag{4.12}$$

where $\nu = \frac{n}{n-1}$. To run the iteration we put $p_0 = 1$ and $p_{k+1} = p_k\nu + \nu$, $k \ge 0$. Hence

$$\begin{aligned} \|w_t\|_{L^{p_{k+1}}} &\le (CC_s)^{\frac{1}{p_k+1}} p_k^{\frac{2}{p_k+1}} \|w\|_{L^{p_k}}^{\frac{p_k}{p_k+1}} \\ &\le (CC_s)^{\sum_{i=0}^{i=k}\frac{\nu^{k-i}}{p_k+1}} \prod_{i=0}^{i=k} p_i^{\frac{2\nu^{k-i}}{p_k+1}} \|w_t\|_{L^1}^{\prod \frac{p_i}{p_i+1}} \\ &\le C(n,g)C_s^{\frac{n}{2}} \|w_t\|_{L^1}^{\gamma(n)} \end{aligned}$$

for a constant $\gamma(n)$ depending only on n, where we have used the fact $p_k \le 2\nu^k$ for $k \ge 1$. Therefore by (4.10), we prove

$$\|w_t\|_{C^0} \le C(n,g)C_s^{\frac{n}{2}} \|w_t\|_{L^1}^{\gamma(n)} \to 0, \text{ as } t \to \infty.$$

□

5. Another version of the invariance $N_X(\omega_g)$

Let $Y \in \eta_r(M)$ so that $\mathrm{Im}(Y)$ generates an one-parameter compact subgroup of K. Denote $\mathcal{K}_Y$ to be a class of K_Y-invariant Kähler metrics in $2\pi c_1(M)$. Then according to the proof of Proposition 1.4, we actually prove

$$\sup_{g\in\mathcal{K}_Y} \lambda(g) \le (2\pi)^{-n}[nV - \tilde{F}_Y(Y) - N_Y(c_1(M))]. \tag{5.1}$$

Note that

$$\tilde{F}_Y(Y) = \int_M Y(h_g - \theta_{Y,\omega})e^{\theta_{Y,\omega_g}}\omega_g^n$$

and

$$N_Y(c_1(M)) = \int_M \theta_{Y,\omega_g} e^{\theta_{Y,\omega_g}}\omega_g^n$$

are both holomorphic invariances of M. In this section, we want to show

Proposition 5.1. *Let $H(Y) = \tilde{F}_Y(Y) + N_Y(c_1(M))$. Then*

$$\sup_{Y\in\eta_r(M)} H(Y) = N_X(c_1(M)),$$

where X is the extremal vector field as in Section 1.

Proof. Choose a constant c_Y so that $\hat{\theta}_{Y,\omega_g} = \theta_{Y,\omega_g} + c_Y$ satisfies a normalization condition

$$\int_M \hat{\theta}_{Y,\omega_g} e^{h_g} \omega_g^n = 0. \tag{5.2}$$

Then $\hat{\theta}_{Y,\omega_g}$ satisfies an equation

$$\Delta \hat{\theta}_{X,\omega_g} + X(h_g) + \hat{\theta}_{X,\omega_g} = 0$$

Thus using the integration by part, we have

$$\tilde{F}_Y(Y) + \int_M \hat{\theta}_{Y,\omega_g} e^{\theta_{Y,\omega_g}} \omega_g^n = 0$$

It follows

$$H(Y) = -c_Y V = \int_M \theta_{Y,\omega_g} e^{h_g} \omega_g^n. \tag{5.3}$$

We compute the first variation of $H(Y)$ in $\eta_r(M)$. By the definition of $\theta_{Y+tY'}$, we see that there exist constants $b(t)$ such that $\theta_{Y+tY'} = \theta_Y + t\theta_{Y'} + b(t)$. Since $\int_M e^{\theta_{Y+tY'}} \omega_g^n = V$, we have

$$e^{-b(t)} = \frac{1}{V} \int_M e^{\theta_Y + t\theta_{Y'}} \omega_g^n.$$

Thus we get

$$\frac{dH(Y+tY')}{dt}|_{t=0} = \int_M \theta_{Y'} e^{h_g} \omega_g^n - \int_M \theta_{Y'} e^{\theta_Y} \omega_g^n = \tilde{F}_Y(Y'). \tag{5.4}$$

Therefore, by [TZhu2], we see that there exists a unique critical $X \in \eta_r(M)$ of $H(\cdot)$ such that

$$\tilde{F}_X(Y') = F_X(Y') \equiv 0, \ \forall \ Y' \in \eta_r(M). \tag{5.5}$$

Similarly, we have

$$\theta_{tY+(1-t)Y'} = t\theta_Y + (1-t)\theta_{Y'} + b(t)', \ \forall \ t \in [0,1]$$

for some constants $b(t)'$. Then

$$\begin{aligned} V = \int_M e^{\theta_{tY+(1-t)Y'}} \omega_g^n &= e^{b(t)'} \int_M e^{t\theta_Y + (1-t)\theta_{Y'}} \omega_g^n \\ &\leq e^{b(t)'} [t \int_M e^{\theta_Y} \omega_g^n + (1-t) \int_M e^{\theta_{Y'}} \omega_g^n] \\ &= e^{b(t)'} V. \end{aligned}$$

Thus $b(t)' \geq 0$. Consequently

$$H(tX + (1-t)Y) \geq tH(X) + (1-t)H(Y).$$

This means that $H(\cdot)$ is a concave functional on $\eta_r(M)$. It follows that X is a global maximizer of $H(\cdot)$. Therefore we prove the proposition by using the fact $H(X) = N_X(c_1(M))$.

$\square$

Corollary 5.2. *Let $\mathcal{K}_K$ be a class of K-invariant Kähler metrics in $2\pi c_1(M)$. Suppose that*

$$\sup_{g\in\mathcal{K}_K} \lambda(g) < \inf_{Y\in\eta_r(M)} (2\pi)^{-n}[nV - F_Y(Y) - N_Y(c_1(M))]. \tag{5.6}$$

Then (M, J) could not admit any Kähler-Ricci soliton. Furthermore, the modified Mabuchi's K-energy could not be bounded from below.

Proof. The first part of corollary follows from Proposition 5.1 and Corollary 1.5. The second part follows from Proposition 5.1 and Theorem 0.1.

$\square$

The above corollary gives a new obstruction to the existence of Kähler-Ricci solitons.

6. Proof of Theorem 0.2

In this section, we will modify the proof of Main Theorem in [TZhu5] to prove Theorem 0.2. The proof in [TZhu5] depends on a generalized uniqueness theorem for Kähler-Einsteins recently proved by Chen and Sun in [CS]. Here we avoid to use their theorem so that we can generalize the proof to the case of Kähler-Ricci solitons by applying Proposition 3.1. As in [TZhu5], we write an initial Kähler form ω_g of Kähler-Ricci flow (2.1) by

$$\omega_g = \omega_\varphi = \omega_{g_{KS}} + \sqrt{-1}\partial\overline{\partial}\varphi \in 2\pi c_1(M)$$

for a Kähler potential φ on M. We define a path of Kähler forms

$$\omega_{g^s} = \omega_{g_{KS}} + s\sqrt{-1}\partial\overline{\partial}\varphi$$

and set

$$I = \{s \in [0,1] | \ (g_t^s; g^s) \text{ converges to a Kähler-Ricci soliton in } C^\infty \\ \text{in sense of Kähler potentials}\}.$$

Clearly, I is not empty by the assumption of existence of Kähler-Ricci solitons on M. We want to show that I is in fact both open and closed. Then it follows that $I = [0, 1]$. This will finish the proof Theorem 0.2.

The openness of I is related to the following stability theorem of Kähler-Ricc flow, which was proved in [Zhu2].

Lemma 6.1. *Let (M, J) be a compact Kähler manifold which admits a Kähler-Ricci soliton (g_{KS}, X). Let ψ be a Kähler potential of a K_X-invariant initial metric g of (2.1). Then there exists a small ϵ such that if*

$$\|\psi\|_{C^3} \leq \epsilon,$$

the solution $g(t, \cdot)$ of (2.1) will converge to a Kähler-Ricci soliton with respect to X in C^∞ in the sense of Kähler potentials. Moreover, the convergence can be made exponentially.

Remark 6.2. *Lemma 6.1 is still true if the K_X-invariant condition is removed for the initial metric g (cf. [Zhu2]). But we do not know whether the convergence is exponentially fast or not.*

Proof of openness of I. Suppose that $s_0 \in I$. Then by the uniqueness of Kähler-Ricci solitons [TZhu1], the flow $(g_t^{s_0}; \omega_{s_0})$ converges to g_{KS} after a holomorphism transformation in $\mathrm{Aut}_r(M)$. Namely, there exists a $\sigma \in \mathrm{Aut}_r(M)$ such that $\sigma^\star \omega_{g_t^{s_0}} = \omega_{KS} + \sqrt{-1}\partial\overline{\partial}(\varphi_t^{s_0})_\sigma$ with property

$$\|(\varphi_t^{s_0})_\sigma\|_{C^k} \leq C_k e^{-\alpha_k t},$$

where $C_k, \alpha_k > 0$ are two uniform constants. Then we can choose T sufficiently large such that

$$\|(\varphi_t^{s_0})_\sigma\|_{C^3(M)} < \frac{\delta}{2},$$

where δ is a small number determined in Lemma 6.1. Since the Kähler-Ricci flow is stable for any fixed finite time, there is a small $\epsilon > 0$ such that

$$\|\varphi_T^s - (\varphi_T^{s_0})_\sigma\|_{C^3(M)} < \frac{\delta}{2}, \ \forall\ s \in [s_0, s_0 + \epsilon],$$

where φ_T^s is a Kähler potential of evolved Kähler metric g_T^s of Kähler-Ricci flow $(g_t^s; \sigma^\star\omega_s)$ at time T. Hence, we have

$$\|\varphi_T^s\|_{C^3(M)} < \delta, \ \forall\ s \in [s_0, s_0 + \epsilon]. \tag{6.1}$$

Then the flow $(g_t; g_T^s)$ with initial g_T^s will converge to a Kähler-Ricci soliton in C^∞ according to Lemma 6.1. This shows $s \in I$ for any $s \in [s_0, s_0 + \epsilon]$

□

Let φ_t^s be a family of Kähler potentials of evolved Kähler metric g_t^s of Kähler-Ricci flow $(g_t^s; \omega_s)$. To make potentials φ_t^s more smaller to control, we need the following lemma, which was proved in [TZhu1].

Lemma 6.3. *Let M be a compact Kähler manifold which admits a Kähler-Ricci soliton (g_{KS}, X). Let φ be a K_X-invariant Kähler potential. Then there exists a unique holomorphism transformation $\sigma \in Aut_r(M)$ such that $\varphi_\sigma \in \Lambda^\perp(\omega_{KS})$ with property*

$$J(\varphi_\sigma) = \inf_{\tau \in Aut_r(M)} J(\varphi_\tau),$$

where $\Lambda^{\perp}(\omega_{KS})$ is an orthogonal space to kernel space of linear operator $(\Delta_{g_{KS}} + X + Id)(\psi)$ and

$$J(\varphi) = -\int_M \varphi e^{\theta_{X,\omega_\varphi}} \omega_\varphi^n + \int_0^1 \int_M \varphi e^{\theta_{X,\omega_{\lambda\varphi}}} \omega_{\lambda\varphi}^n \wedge d\lambda \geq 0.$$

Moreover,

$$\|\sigma - Id\| \leq C(\|\varphi\|_{C^5}),$$

where $\|\sigma - Id\|$ denotes the distance norm in Lie group $Aut_r(M)$.

Proof of closedness of I. By the openness of I, we see that there exists a $\tau_0 \leq 1$ with $[0, \tau_0) \subset I$. We need to show that $\tau_0 \in I$. In fact we want to prove that for any $\delta > 0$ there exists a large T such that

$$\|(\phi_t^s)_{\sigma_{s,t}}\|_{C^5} \leq \delta, \ \forall \ t \geq T \text{ and } s < \tau_0, \tag{6.2}$$

where $\sigma_{s,t}$ are some holomorphisms in $\mathrm{Aut}_r(M)$. We will use an argument by contradiction as in [TZhu5]. On contrary, we can find a sequence of evolved Kähler metrics $g_{t_i}^{s_i}$ of Kähler-Ricci flows $(g_t^{s_i}; g^{s_i})$, where $s_i \to \tau_0$ and $t_i \to \infty$, and a sequence of unique holomorphisms $\sigma_{s_i,t_i} \in \mathrm{Aut}_r(M)$ for pairs (s_i, t_i) such that $(\phi_{t_i}^{s_i})_{\sigma_{s_i,t_i}} \in \Lambda^{\perp}(\omega_{KS})$ and

$$\|(\phi_{t_i}^{s_i})_{\sigma_{s_i,t_i}}\|_{C^5} \geq \delta_0 > 0, \tag{6.3}$$

for some constant δ_0. Since the Kähler-Ricci flow $(g_t^{s_i}; g^{s_i})$ converges to some Kähler-Ricci soliton, by the uniqueness of Kähler-Ricci solitons [TZhu1], the flow after a holomorphism transformation in $\mathrm{Aut}_r(M)$ converges to g_{KS}. So we may further assume that $\phi_{t_i}^{s_i}$ satisfy

$$\|(\phi_{t_i}^{s_i})_{\sigma_{s_i,t_i}}\|_{C^5} \leq 2\delta_0. \tag{6.4}$$

Then there exists a subsequence $(\phi_{t_i}^{s_i})_{\sigma_{s_i,t_i}}$ (still used by $(\phi_{t_i}^{s_i})_{\sigma_{s_i,t_i}}$) of $(\phi_{t_i}^{s_i})_{\sigma_{s_i,t_i}}$ converging to a potential $\phi_\infty \in \Lambda^{\perp}(\omega_{KS})$ with property

$$2\delta_0 \geq \|\phi_\infty\|_{C^5} \geq \delta_0. \tag{6.5}$$

We want to show that

$$\lambda(\omega_{\phi_\infty}) = \lambda(g_{KS}) = (2\pi)^{-n}(nV - N_X(c_1(M)). \tag{6.6}$$

First we note that the modified K-energy is bounded from below since M admits a Kähler-Ricci soliton [TZhu2]. Then by Proposition 3.1 and the monotonicity of $\lambda(g_t^{\tau_0})$, we see that for any $\epsilon > 0$, there exists a large $T > 0$ such that

$$\lambda(g_t^{\tau_0}) \geq (2\pi)^{-n}(nV - N_X(c_1(M)) - \frac{\epsilon}{2}, \ \forall \ t \geq T.$$

Since Kähler-Ricci flow is stable in finite time and $\lambda(g_t^s)$ is monotonic in t, there is a small $\delta > 0$ such that for any $s \geq \tau_0 - \delta$, we have

$$\lambda(g_t^s) \geq (2\pi)^{-n}(nV - N_X(c_1(M)) - \epsilon, \ \forall \ t \geq T. \tag{6.7}$$

Since $s_i \to \tau_0$ and $t_i \to \infty$, we conclude that

$$\lim_{s_i\to\tau_0, t_i\to\infty} \lambda(\sigma^\star_{s_i,t_i} g^{s_i}_{t_i}) = \lim_{s_i\to\tau_0, t_i\to\infty} \lambda(g^{s_i}_{t_i}) = (2\pi)^{-n}(nV - N_X(c_1(M)).$$

By the continuity of $\lambda(\cdot)$, we will get (6.6).

Now by Corollary 1.4 together with (6.6), we see that ω_{ϕ_∞} is a global maximizer of $\lambda(\cdot)$ in $\mathcal{K}_X$, so it is a critical point of $\lambda(\cdot)$. Then it is easy to show that ω_{ϕ_∞} a Kähler-Ricci soliton with respect to X by computing the first variation of $\lambda(\cdot)$ as done in (1.5). Thus by the uniqueness result for Kähler-Ricci solitons in [TZhu1], we get

$$\omega_{\phi_\infty} = \sigma^\star \omega_{KS},$$

where $\sigma \in \mathrm{Aut}_r(M)$. Since $\phi_\infty \in \Lambda^\perp(\omega_{KS})$, by Lemma 6.3, ϕ_∞ must be zero. This is a contradiction to (6.5). The contradiction implies that (6.2) is true.

By (6.2), we see that for any $\delta > 0$ there exists a large T_0 and $\sigma_0 \in \mathrm{Aut}_r(M)$ such that

$$\|(\phi^{\tau_0}_{T_0})_{\sigma_0}\|_{C^5} \le \delta. \tag{6.8}$$

Then by Lemma 6.1, the Kähler-Ricci flow $(g_t; \omega_{(\phi^{\tau_0}_{T_0})_{\sigma_0}})$ converge to a Kähler-Ricci soliton. Thus, we prove that $\tau_0 \in I$.

$\square$

Remark 6.4. *According to the proof of Theorem 0.2 and Remark 6.2, Theorem 0.2 will be still true if the K_X-invariant condition is removed for the initial metric g of (2.1) assuming that Conjecture (3.3) is true.*

7. Appendix

In [TZhu5], it was proved the minimizer f_t of $W(g_t, \cdot)$-functional associated to evolved Kähler metric g_t of Kähler-Ricci flow (2.1) is uniformly bounded (see also [TZha]). In this appendix, we show that the gradients of f_t are also uniformly bounded, and so are $\triangle f_t$ by (1.4). Namely, we prove

Theorem 7.1. *There is a uniform constant C such that*

$$\|f_t\| + \|\nabla f_t\| + \|\triangle f_t\| \le C, \ \forall\ t > 0.$$

We will derive $\|\nabla f_t\|$ in Theorem 7.1 by studying a general nonlinear elliptic equation as follows:

$$\triangle w(x) = w(x) F(x, w(x)) \tag{7.1}$$

where the Laplace operator $\triangle$ is associated to a Kähler metric g in $2\pi c_1(M)$ and F is a smooth function on $M \times \mathbb{R}^+$, which satisfies a structure condition:

$$-A - Bt^\alpha \le F(\cdot, t) \le H(t). \tag{7.2}$$

Here $0 \le A, B \le \infty, 0 \le \alpha < \frac{2}{n}$ are constants, and H is a proper function on $\mathbb{R}^+$ which satisfies a growth control at 0:

$$\limsup_{t\to 0} \big(tH(t)\big) < \infty. \tag{7.3}$$

Lemma 7.2. *Let w is a positive solution of (7.1). Then*

$$\|\nabla w\|_{C^0} \le C(n) C_s^{\frac{n}{2}} \big(\|\nabla h\|_{C^0} + \|wF\|_{C^0}\big)^n \Big(\int_M (1+|\nabla w|^2) dV_g\Big)^{1/2}, \tag{7.4}$$

where C_s is a Sobelev constant of g and h is a Ricci potential of g.

Proof. We will use the Moser's iteration to L^p-estimate of $|\nabla w|$. By the Bochner formula, we have

$$\begin{aligned}\triangle|\nabla w|^2 &= |\nabla\nabla w|^2 + |\nabla\bar{\nabla} w|^2 + \nabla_i \triangle w \nabla_{\bar{i}} w + \nabla_i w \nabla_{\bar{i}} \triangle w + R_{i\bar{j}} \nabla_{\bar{i}} w \nabla_j w\\ &= |\nabla\nabla w|^2 + |\nabla\bar{\nabla} w|^2 + \nabla_i (wF) \nabla_{\bar{i}} w + \nabla_i w \nabla_{\bar{i}} (wF) + R_{i\bar{j}} \nabla_{\bar{i}} w \nabla_j w.\end{aligned}$$

Put $\eta = |\nabla w|^2 + 1$. Then for $p \ge 2$, it follows

$$\begin{aligned}&\frac{4(p-1)}{p^2} \int_M |\nabla \eta^{p/2}|^2 dV_g\\ &= -\int_M \eta^{p-1} \triangle \eta dV_g\\ &= -\int_M \eta^{p-1} \big(|\nabla\nabla w|^2 + |\nabla\bar{\nabla} w|^2\big) dV_g - \int_M \eta^{p-1} R_{i\bar{j}} \nabla_{\bar{i}} w \nabla_j w dV_g\\ &- \int_M \eta^{p-1} \big(\nabla_i (wF) \nabla_{\bar{i}} w + \nabla_{\bar{i}} (wF) \nabla_i w\big) dV_g.\end{aligned} \tag{7.5}$$

The last term on the right hand side can be estimate as follows.

$$\begin{aligned}&- \int_M \eta^{p-1} \big(\nabla_i (wF) \nabla_{\bar{i}} w + \nabla_{\bar{i}} (wF) \nabla_i w\big) dV_g\\ &= \int_M wF \big(\nabla_i \eta^{p-1} \nabla_{\bar{i}} w + \nabla_{\bar{i}} \eta^{p-1} \nabla_i w + 2\eta^{p-1} \triangle w\big) dV_g\\ &= \frac{2(p-1)}{p} \int_M wF \eta^{\frac{p}{2}-1} \big(\nabla_i \eta^{p/2} \nabla_{\bar{i}} w + \nabla_{\bar{i}} \eta^{p/2} \nabla_i w\big) dV_g\\ &+ 2\int_M wF \eta^{p-1} \triangle w dV_g.\end{aligned}$$

Then

$$
\begin{aligned}
& -\int_M \eta^{p-1}\big(\nabla_i(wF)\nabla_{\bar i}w dV_g + \nabla_{\bar i}(wF)\nabla_i w\big)dV_g \\
& \le \frac{2(p-1)}{p^2}\int_M |\nabla\eta^{p/2}|^2 dV_g + 2(p-1)p\int_M (wF)^2\eta^{p-2}(\eta-1)dV_g \\
& + \int_M \eta^{p-1}\Big(\frac{(\triangle w)^2}{2n} + 2n(wF)^2\Big)dV_g \\
& \le \frac{2(p-1)}{p^2}\int_M |\nabla\eta^{p/2}|^2 dV_g + \frac{1}{2n}\int_M \eta^{p-1}(\triangle w)^2 dV_g \\
& + 2[p(p-1)+n]\|wF\|^2_{C^0}\int_M \eta^p dV_g.
\end{aligned}
\tag{7.6}
$$

For the second term on the right hand side, we note

$$R_{i\bar j} = g_{i\bar j} + h_{i\bar j}.$$

Then

$$
\begin{aligned}
& -\int_M \eta^{p-1}R_{i\bar j}\nabla_{\bar i}w\nabla_j w dV_g \\
& = \int_M \eta^{p-1}\nabla_i\nabla_{\bar j}h\nabla_{\bar i}w\nabla_j w dV_g - \int_M \eta^p dV_g \\
& = \frac{2(p-1)}{p}\int_M \eta^{p/2-1}\nabla_{\bar j}h\nabla_i\eta^{p/2}\nabla_{\bar i}w\nabla_j w dV_g \\
& + \int_M \eta^{p-1}\nabla_{\bar j}h\big(\triangle w\nabla_j w + \nabla_{\bar i}w\nabla_i\nabla_j w\big)dV_g - \int_M \eta^p dV_g.
\end{aligned}
$$

Thus

$$
\begin{aligned}
& -\int_M \eta^{p-1}R_{i\bar j}\nabla_{\bar i}w\nabla_j w dV_g \\
& \le \frac{p-1}{p^2}\int_M |\nabla\eta^{p/2}|^2 dV_g + p(p-1)\|\nabla h\|^2_{C^0}\int_M \eta^{p-2}(\eta-1)^2 dV_g \\
& + \frac{1}{2n}\int_M \eta^{p-1}(\triangle w)^2 dV_g + \frac{n}{2}\|\nabla h\|^2_{C^0}\int_M \eta^{p-1}(\eta-1)dV_g \\
& + \frac{1}{2n}\int_M \eta^{p-1}|\nabla\nabla w|^2 dV_g + \frac{n}{2}\|\nabla h\|^2_{C^0}\int_M \eta^{p-1}(\eta-1)dV_g \\
& \le \frac{p-1}{p^2}\int_M |\nabla\eta^{p/2}|^2 dV_g + \frac{1}{2n}\int_M \eta^{p-1}(\triangle w)^2 dV_g \\
& + \frac{1}{2n}\int \eta^{p-1}|\nabla\nabla w|^2 dV_g + [p(p-1)+n]\|\nabla h\|^2_{C^0}\int_M \eta^p dV_g.
\end{aligned}
\tag{7.7}
$$

Substituting (7.6) and (7.7) into (7.5), we get

$$\begin{aligned}
&\frac{p-1}{p^2}\int_M |\nabla\eta^{p/2}|^2 dV_g \\
&\leq -\int_M \eta^{p-1}\big(|\nabla\nabla w|^2 + |\nabla\bar{\nabla} w|^2\big) dV_g \\
&+ \frac{1}{n}\int_M \eta^{p-1}(\triangle w)^2 dV_g + \frac{1}{2n}\int_M \eta^{p-1}|\nabla\nabla w|^2 dV_g \\
&+ [2(p-1)p+n]\big(\|\nabla h\|^2_{C^0} + \|wF\|^2_{C^0}\big)\int_M \eta^p dV_g \\
&\leq C(n)p^2\big(\|\nabla h\|^2_{C^0} + \|wF\|^2_{C^0}\big)\int_M \eta^p dV_g.
\end{aligned}$$

It follows

$$\int_M |\nabla\eta^{p/2}|^2 dV_g \leq C(n)p^3\big(\|\nabla h\|^2_{C^0} + \|wF\|^2_{C^0}\big)\int_M \eta^p dV_g, \quad \forall\ p \geq 2.$$

Therefore, by iteration, we derive

$$\sup \eta \leq C(n)D^{n/2}\Big(\int_M \eta^2 dV_g\Big)^{1/2},$$

where $D = C_s\big(\|\nabla h\|^2_{C_0} + \|wF\|^2_{C^0}\big)$. This implies (7.4) . □

Proposition 7.3.

$$\|\nabla w\|_{C^0} \leq C(\|w\|_{L^2}), \tag{7.8}$$

where the constant C depends only on n, C_s, A, B, α, H, Vol(g), $\|\nabla h\|_{C^0}$ and $\|w\|_{L^2}$.

Proof. First we note that by using the standard Moser's iteration to equation

$$\triangle w(x) \geq -A - Bw^\alpha,$$

it is easy to see

$$\sup w \leq C(1 + \|w\|^\gamma_{L^2})$$

for some constants C and γ which depend only on n, C_s, A, B, α, H and Vol(g). On the other hand, by (7.1), we have

$$\int_M |\nabla w|^2 dV_g = -\int_M w\triangle w dV_g = -\int_M wF dV_g.$$

Then we see that $\|\nabla w\|_{L^2}$ is bounded by $\|w\|_{L^2}$. Thus the proposition follows from Lemma 7.2.

□

Since $v_t = e^{-\frac{f_t}{2}}$ satisfies (3.10) which is a type of equation (7.1), then by Perelman's estimates (d) in Lemma 2.1 and Zhang's estimate for Sobelev constants associated to g_t in [Zha] together with C^0-estimate for f_t in [TZhu5], we obtain a uniform gradient estimate for v_t from Proposition 7.3, and so for f_t . By equation (1.4), we also derive a a uniform Laplacian estimate for f_t. Thus Theorem 7.1 is true. Theorem 7.1 will be used in Section 4.

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